INTERNATIONAL MATHEMATICS
TOURNAMENT OF TOWNS

JUNIOR PAPER: YEARS 8,9,10

Tournament 40, Northern Autumn 2018 (A Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. In triangle ABC, M is the midpoint of the side BC and E is a point on the side AC distinct from A and C. Suppose that $BE \geq 2AM$. Prove that one of the angles of triangle ABC is obtuse. (5 points)

2. There are 2018 people living on an island. Each person is one of: a knight, a knave, or a neither-knight-nor-knave. A knight always tells the truth, and a knave always lies. A neither-knight-nor-knave answers as the majority of people answered before him, or randomly, in the case that the numbers of *"Yes"* and *"No"* answers are equal. Everyone on the island knows which of the three possibilities each person is. One day all 2018 inhabitants of the island were arranged in a line and each in turn answered *"Yes"* or *"No"* to the same question:

Are there more knights than knaves on the island?

The total number of *"Yes"* answers was 1009 and everyone heard all the previous answers. Determine the maximum possible number of neither-knight-nor-knave people among the inhabitants of the island. (6 points)

3. One needs to write a number of the form $77\ldots7$, in base ten, using only 7s, the operations of addition, subtraction, multiplication, division, and raising to a power, and brackets. One can also use any number of 7s together with no operations between them. For the number 77 the shortest way to write it is to simply write 77. Does there exist a number of the form $77\ldots7$ that can be written under the rules above using a smaller number of 7s than in its base ten notation? (8 points)

4. A 7×7 grid board can be empty or can contain an invisible 2×2 ship that is located with its edges along the grid lines. A detector placed in a square of the board shows whether or not the square is occupied by the ship. All the detectors on the board are to be switched on at the same time. What is the smallest number of detectors needed to determine if the ship is on the board and, if so, exactly where it is located? (8 points)

Continued over the page

5. Let $ABCD$ be an isosceles trapezium (with AD parallel to BC), that is inscribed in a circle with centre O. The line BO and side AD meet at the point E. Suppose that O_1 and O_2 are the circumcentres of the triangles ABE and DBE, respectively. Prove that the points O_1, O_2, O and C are concyclic. (8 points)

6. Prove that

(a) any integer of the form $3k-2$, where k is an integer, can be represented as the sum of a perfect square and two perfect cubes of some integers. (7 points)

(b) any integer can be represented as the sum of a perfect square and three perfect cubes of some integers. (3 points)

7. There are $n \geq 2$ towns in some virtual world. Some pairs of towns are connected by roads, but there is no more than one road between any pair of towns. Any town can be reached from any other town via the roads. One can change a road only in a town. The world is called *simple*, if it is impossible to start at some town and to return to that same town without using the same road twice. Otherwise the world is called *complex*. Petya and Vasya play the following game:

> At the start Petya chooses a single direction on each road so that the road can be used in the chosen direction only and places a virtual tourist in one of the towns. Then Petya moves the tourist along a road in the permitted direction to a neighbouring town. On his turn, Vasya changes the permitted direction on one of the inbound or outbound roads of the town where the tourist is at the moment. Vasya wins if Petya cannot make a move.

Prove that

(a) in a simple world Petya can avoid defeat no matter how Vasya plays. (5 points)

(b) in a complex world Vasya can win for sure no matter how Petya plays. (7 points)